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On the absolute approximation ratio for First Fit and related results

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ABSTRACT

We revisit three famous bin packing algorithms, namely Next Fit (NF), Worst Fit (WF) and First Fit (FF).

We compare the approximation ratio of these algorithms as a function of the total size of the input items α. We give a complete analysis of the worst case behavior of WF and NF, and determine the ranges of α for which FF has a smaller approximation ratio than WF and NF.

In addition, we prove a new upper bound of $\frac{12}{7} \approx 1.7143$ on the absolute approximation ratio of FF, improving over the previously known upper bound of 1.75, given by Simchi-Levi. This property of FF is in contrast to the absolute approximation ratios of WF and NF, which are both equal to 2.

1. Introduction

We consider the classical bin packing problem where we are given an unlimited number of unit sized bins and a sequence of items each with a positive size. The goal is to pack the items in the bins so that the total size of the items in each bin is at most 1, while minimizing the number of bins used to pack all the items. The total size of items packed into a bin is also called its *load*. We say that a bin is *open* if it has non-zero load. The action of starting the usage of an empty bin is called *opening* a new bin. The ordering of the bins corresponds to the order in which they were opened.

We consider the approximation ratio as a function of the total size of the input items, denoted by α. Thus we compare the cost of an algorithm $\mathcal{A}$ to an optimal solution Opt, but we restrict this comparison to sub-classes of inputs. The (worst case) approximation ratio of an algorithm, $\mathcal{A}$, for α is

$$\mathcal{R}_{\mathcal{A}}(\alpha) = \max_{I:|I|=\alpha} \frac{\mathcal{A}(I)}{\textsc{Opt}(I)}$$

where $|I|$ denotes the sum of the sizes of the items in the sequence I, and $\mathcal{A}(I)(\textsc{Opt}(I))$ denotes the number of bins used by algorithm $\mathcal{A}(\textsc{Opt})$ on sequence I. The absolute approximation ratio of an algorithm $\mathcal{A}$ can be defined as

$$\mathcal{R}_{\mathcal{A}} = \sup_{\alpha>0} \mathcal{R}_{\mathcal{A}}(\alpha),$$

where this definition is equivalent to the standard definition, $\mathcal{R}_{\mathcal{A}} = \sup_I \mathcal{A}(I)/\textsc{Opt}(I)$.

Note that a more standard way to parameterize worst case ratios is by the cost of Opt, as follows: $R_{\mathcal{A}}(n) = \max_{I:OPT(I)=n}(\mathcal{A}(I)/\textsc{Opt}(I))$. According to this definition, the absolute approximation ratio of an algorithm is defined as

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$\sup_{n>0} \mathcal{R}_{\mathcal{A}}(n)$, and the asymptotic approximation ratio of an algorithm is defined as $\limsup_{n\to\infty} \mathcal{R}_{\mathcal{A}}(n)$. It is not difficult to see that for a given input I, $|I| \leq \text{OPT}(I) \leq 2|I|$, and thus an equivalent definition of the asymptotic approximation ratio is $\limsup_{\alpha\to\infty} \mathcal{R}_{\mathcal{A}}(\alpha)$. Note also that if the algorithms are online, the (asymptotic) approximation ratio is also known as the (asymptotic) competitive ratio (see [10]).

We next define the algorithms studied in this paper. These algorithms are online, in the sense that the items are packed in the order given by the input list, and differ as to the packing rule used for packing the items. The packing rules for the three algorithms studied are as follows.

- First-Fit (FF) [11,8]: If there is any open bin in which the current item would fit, place the item in the first such bin, and otherwise open a new bin and place the item there.
- Worst-Fit (WF) [7]: If there is any open bin in which the current item would fit, place the item in a least loaded such bin, and otherwise open a new bin and place the item there.
- Next-Fit (NF) [7]: Place the current item in the last open bin if it would fit into it, and otherwise open a new bin and place the item there.

The asymptotic competitive ratio of FF is 1.7 [8], while WF's asymptotic competitive ratio is 2 [7], and the asymptotic competitive ratio for NF is also 2 [7]. Many other performance measures for bin packing algorithms separate FF and WF, showing that FF is the better algorithm. Several measures (but not all) are also known to show that WF is better than NF [1,2,4,5,3] (for some measures it is unknown whether they separate these two algorithms, while the asymptotic competitive ratio does not separate them). Thus, we find these algorithms interesting in that, although it is intuitively expected that First-Fit will most often perform best and Next-Fit will most often perform worst in terms of minimizing the number of bins used, not all quality measures reflect this.

The class of Any-Fit algorithms consists of all algorithms which never open a new bin if the current item can fit into an open bin. Clearly, FF and WF belong to this class while NF does not. The statement above concerning intuition is vague, but it can be made very concrete for Next-Fit compared with the other algorithms, since on any given input sequence of items, Next-Fit will use at least as many bins as any Any-Fit algorithm. In particular, the next proposition (which is folklore) strengthens the comparison between NF and WF (and with other Any-Fit algorithms).

Proposition 1. NF *will use at least as many bins as an Any-Fit algorithm A on any sequence of items.*

Proof. Given an input, let $B_{\text{NF}}(i)$ denote the bin number where NF places the item that algorithm A places as the first in bin i. We show by induction on i that $B_{\text{NF}}(i) \geq i$ for all i. Both values are 1 for $i = 1$. Suppose it holds for some value t. Then A opens a new bin, t, with item j and NF places j in some bin $t' \geq t$. Consider the item, k, where A opens bin $t+1$. If NF has not already opened bin $t'+1$ (before processing item k), then it has packed all items between items j and k (including item j but excluding item k) into bin t'. When A opened bin $t+1$, the contents of its bin t must have been a subset of these items. Since item k did not fit with this subset, it could not have fit with the whole set either, and hence under NF it must be packed into some bin $t'' > t' \geq t$. Hence $B_{\text{NF}}(t+1) \geq t+1$, and by induction $B_{\text{NF}}(i) \geq i$ for all i. Thus, NF uses at least as many bins as A. $\square$

Even though $\mathcal{R}_{\text{FF}} < \mathcal{R}_{\text{WF}}$, an analogue of Proposition 1 does not hold for FF and WF. There exist lists with arbitrarily large values of α for which FF uses $\frac{9}{8}$ times as many bins as WF. For example, consider a list consisting of five batches of items, arriving one after the other in the specified order, where there are $4n$ items of each type. Items of type J_1 are of size $\frac{1}{2}+3\varepsilon$. Items of type J_2 are of size $\frac{1}{2}+2\varepsilon$. Items of type J_3 are of size $\frac{1}{4}-\varepsilon$. Items of type J_4 are of size $\frac{1}{2}-3\varepsilon$ and items of type J_5 are of size $\frac{1}{4}-\varepsilon$ (for some $\varepsilon < \frac{1}{20}$). Under any algorithm, the items of types J_1 and J_2 are assigned one per bin. Under WF, the items of type J_3 are packed into the bins containing items of type J_2 (one item of type J_3 per bin). Then the items of type J_4 are assigned (one item of type J_4 per bin) into bins with items of type J_1, and, finally, the items of type J_5 are assigned (one per bin) into the bins now containing one item of type J_2 and one of type J_3, completely filling those bins, for a total of $8n$ bins. Under FF, the items of type J_3 are assigned (one per bin) into the bins containing items of type J_1. Then the items of type J_4 are assigned (one per bin) into the bins containing items of type J_2, and, finally, the items of type J_5, which no longer fit in any of the partially-filled bins, are packed into new bins, using n bins, for a total of $9n$ bins.

On the other hand, there exist very simple lists with arbitrarily large values of α, for which WF uses more bins than FF, so the intuition regarding FF being a better algorithm than WF is not wrong. One such example is a list which consists of three batches of items, where the items of the first type are of size 0.6 ($8n$ such items), the items of the second type are of size 0.1 ($8n$ such items), and the items of the third type are of size 0.4 ($6n$ such items). It is not difficult to see that WF creates $11n$ bins, while FF finds an optimal solution which uses $8n$ bins.

The interest in the absolute approximation ratio of bin packing algorithms comes from the fact that this measure is much more common in general, while bin packing is historically studied using the asymptotic approximation ratio, comparing the heuristic solution to an optimal solution for inputs of a large optimal cost. Johnson et al. [8] give an example where FF uses 17 bins, while an optimal solution can pack the same items into 10 bins, proving a lower bound of 1.7 on the absolute approximation ratio of FF. On the other hand, Simchi-Levi [9] showed an upper bound of 1.75 on the absolute approximation ratio of FF.

In Section 2, we present a complete study of WF and NF as a function of α. We show that the approximation ratio of NF is $2-\frac{1}{\lceil\alpha\rceil}$, while the approximation ratio of WF is $\frac{\lceil 2\alpha\rceil-1}{\lceil\alpha\rceil}$, if $\alpha > 1$ (both approximation ratios are equal to 1 for $\alpha \leq 1$). Thus, if $\alpha \in (j, j+\frac{1}{2}]$ for some integer $j > 0$, then the approximation ratios of NF and WF are $\frac{2j+1}{j+1}$ and $\frac{2j}{j+1}$, respectively. In the case where $\alpha \in (j+\frac{1}{2}, j+1]$ for some integer $j > 0$, both approximation ratios are $\frac{2j+1}{j+1}$.

In Section 3, we present bounds on $\mathcal{R}_{\mathrm{FF}}(\alpha)$ and compare them to those for WF and NF, showing that WF's approximation ratio is larger except in the ranges of α where their approximation ratios are equal. These ranges are: $\alpha \in (0, 2.5] \cup (2.75, 3] \cup (3.25, 3.5]$.

Section 4 presents our main result, which improves the known upper bound on the absolute approximation ratio of FF from 1.75 to $\frac{12}{7} \approx 1.714$. This was independently proven in [12].

Section 5 summarizes our results and presents some remaining open problems.

2. The performance of WF and NF as a function of α

Let α denote the total size of items in the input, and let $W_\alpha = \lceil \alpha \rceil$. Clearly, OPT $\geq W_\alpha$ and $\alpha \leq W_\alpha < \alpha + 1$. Let $W'_\alpha = \frac{\lceil 2\alpha \rceil}{2}$. Note that $W'_\alpha \geq \alpha$; it is not necessarily integral, though $2W'_\alpha$ is an integer. Moreover, $2W'_\alpha < 2\alpha + 1$, i.e., $W'_\alpha < \alpha + \frac{1}{2}$.

Note that if $\alpha \in (j, j+\frac{1}{2}]$ for some integer j, then $W_\alpha = j+1$ while $W'_\alpha = j + \frac{1}{2} = W_\alpha - \frac{1}{2}$. In this case $2W'_\alpha$ is odd. If $\alpha \in (j+\frac{1}{2}, j+1]$ for some integer j, then $W_\alpha = W'_\alpha = j+1$. In this case $2W'_\alpha$ is even. Note that in all cases we have $\alpha \leq W'_\alpha \leq W_\alpha$.

We use the following simple property, which holds by the definitions of NF, and of ANY-FIT algorithms.

Observation 2. (i) *Consider an output of* NF. *The total size of the items in any pair of consecutive bins exceeds* 1.
(ii) *Consider an output of an* ANY-FIT *algorithm* $\mathcal{A}$. *The total size of the items in any pair of (not necessarily consecutive) bins exceeds* 1.

Corollary 3. *Consider the packing of an* ANY-FIT *algorithm* $\mathcal{A}$ *and consider some number* $\Delta \geq 2$ *of (not necessarily consecutive) bins used by this packing. Then the total size of items packed in these bins exceeds* $\frac{\Delta}{2}$. *In particular, if all bins used by* $\mathcal{A}$ *are considered, then* $\alpha > \frac{\Delta}{2}$.

Let Γ *be the number of bins used by* NF. *Then* $\alpha > \lfloor \frac{\Gamma}{2} \rfloor$.

Proof. To prove the first property, we consider all ordered pairs of bins in the subset of Δ bins, such that every bin participates in $\Delta - 1$ pairs. We have $\Delta(\Delta-1)$ pairs, and therefore a total size of more than $\frac{\Delta(\Delta-1)}{2}$ taking all the items in all pairs into account. Since each bin was considered exactly $\Delta - 1$ times, we get a total size which exceeds $\frac{\Delta}{2}$. If the entire set of bins is considered, then we get $\alpha > \frac{\Delta}{2}$.

The second property follows from partitioning the output of NF (except for the last bin, if Γ is odd) into pairs of consecutive bins. $\square$

If $\alpha \leq 1$, these algorithms use only a single bin, so the approximation ratio is 1. Hence, we assume in what follows that $\alpha > 1$.

Lemma 4. *The number of bins used by* NF *is at most* $2\lceil \alpha \rceil - 1 = 2W_\alpha - 1$, *this bound is tight, and the approximation ratio of* NF *satisfies* $\mathcal{R}_{\mathrm{NF}}(\alpha) = 2 - \frac{1}{W_\alpha} = 2 - \frac{1}{\lceil \alpha \rceil}$.

Proof. For an upper bound, assume by contradiction that NF uses at least $2W_\alpha$ bins. By Corollary 3, we get $W_\alpha \geq \alpha > \lfloor \frac{2W_\alpha}{2} \rfloor = W_\alpha$ (since W_α is integral), which is a contradiction. To show that this bound is tight, let $\varepsilon = \frac{\alpha+1-W_\alpha}{W_\alpha}$. Since $W_\alpha < \alpha + 1$, then $\varepsilon > 0$. Consider an input which contains $W_\alpha - 1$ times the pair $\langle \varepsilon, 1 \rangle$, and an additional item of size ε. The number of used bins is $2W_\alpha - 1$, and the total size of items is $W_\alpha - 1 + W_\alpha \varepsilon = W_\alpha - 1 + \alpha + 1 - W_\alpha = \alpha$. An optimal solution packs each item of size 1 in a bin, and all additional items, whose total size is $\alpha + 1 - W_\alpha \leq 1$ (since $W_\alpha \geq \alpha$), in another bin, for a total of W_α bins. $\square$

Lemma 5. *The number of bins used by* WF *is at most* $\lceil 2\alpha \rceil - 1 = 2W'_\alpha - 1$, *this bound is tight, and the approximation ratio of* WF *satisfies* $\mathcal{R}_{\mathrm{WF}} = \frac{2W'_\alpha - 1}{W_\alpha} = \frac{\lceil 2\alpha \rceil - 1}{\lceil \alpha \rceil}$. *The upper bounds hold for any* ANY-FIT *algorithm (but they are not tight for* FF*).*

Proof. For an upper bound, assume for the sake of contradiction that an ANY-FIT algorithm $\mathcal{A}$ uses at least $2W'_\alpha$ bins. By Corollary 3, we get $W'_\alpha \geq \alpha > \frac{2W'_\alpha}{2} = W'_\alpha$, which is a contradiction. To show that this bound is tight for WF, let $\varepsilon = \frac{\alpha + \frac{1}{2} - W'_\alpha}{2W'_\alpha - 1}$. Since $W'_\alpha < \alpha + \frac{1}{2}$, then $\varepsilon > 0$. Consider an input which contains $2W'_\alpha - 1$ times the pair $\langle \frac{1}{2}, \varepsilon \rangle$. Every bin will contain a pair of such items. Every item of size $\frac{1}{2}$ requires a new bin, and for every item of size ε, the last bin is the worst fit. Therefore, the number of used bins is $2W'_\alpha - 1$, and the total size of the items is $(2W'_\alpha - 1)(\frac{1}{2} + \varepsilon) = \frac{2W'_\alpha - 1}{2} + \alpha + \frac{1}{2} - W'_\alpha = \alpha$. If $2W'_\alpha$ is even, then an optimal solution packs $W'_\alpha - 1$ pairs of items of size $\frac{1}{2}$ in $W'_\alpha - 1$ bins, and an additional bin contains all remaining items, which have total size $\alpha - W'_\alpha + 1 \leq 1$. This gives a total of $W'_\alpha = W_\alpha$ bins. If $2W'_\alpha$ is odd, then an optimal solution packs $W'_\alpha - \frac{1}{2}$ pairs of items of size $\frac{1}{2}$ in $W'_\alpha - \frac{1}{2}$ bins, and an additional bin contains all remaining items, which have total size $\alpha - W'_\alpha + \frac{1}{2} \leq \frac{1}{2}$. This gives a total of $W'_\alpha + \frac{1}{2} = W_\alpha$ bins. $\square$

Table 1
Upper bounds on approximation ratios as a function of α. If the column "$\mathcal{R}_{FF}(\alpha) < \mathcal{R}_{WF}(\alpha)$?" is "No", then FF and WF have the same approximation ratio for the given interval of values of α. The bounds for $\alpha \le 1$ are immediate.

Range of α	$\mathcal{R}_{FF}(\alpha)$	$\mathcal{R}_{WF}(\alpha)$	$\mathcal{R}_{NF}(\alpha)$	$\mathcal{R}_{FF}(\alpha) < \mathcal{R}_{WF}(\alpha)$?	FFbound proved in
(0, 1]	1	1	1	*No*	
(1, 1.5]	1	1	1.5	*No*	Theorem 6
(1.5, 2]	$3/2 = 1.5$	3/2	3/2	*No*	Theorem 6
(2, 2.5]	$4/3 \approx 1.33$	4/3	5/3	*No*	Theorem 6
(2.5, 2.75]	$\le 4/3 \approx 1.33$	5/3	5/3	Yes	Corollary 12
(2.75, 3]	$5/3 \approx 1.67$	5/3	5/3	*No*	Theorem 6
(3, 3.25]	$\le 5/4 = 1.25$	3/2	7/4	Yes	Corollary 12
(3.25, 3.5]	$3/2 = 1.5$	3/2	7/4	*No*	Theorem 6
(3.5, 4]	$\le 3/2 = 1.5$	7/4	7/4	Yes	Theorem 15
(4, 4.5]	$\le 7/5 = 1.4$	8/5	9/5	Yes	Theorem 15
(4.5, 5]	$\le 8/5 = 1.6$	9/5	9/5	Yes	Theorem 15
(5, 5.5]	$\le 3/2 = 1.5$	5/3	11/6	Yes	Theorem 15
(5.5, 6]	$\le 5/3 \approx 1.67$	11/6	11/6	Yes	Theorem 15
(6, 6.5]	$\le 11/7 \approx 1.5714$	12/7	13/7	Yes	Theorem 15
(6.5, ∞)	$\le 12/7 \approx 1.714$	$\ge 7/4$	$\ge 13/7$	Yes	Section 4

3. The performance of FF as a function of α

In this section, we provide bounds on the value of $\mathcal{R}_{FF}(\alpha)$ as a function of α. Table 1 summarizes our results, and compares them to those for NF and WF, as derived from Lemmas 4 and 5 of the previous section.

In order to compare FF's approximation ratio with WF's, we first consider the cases where the performance of FF turns out to be the same as WF. These are the cases $\alpha \in (1, 2.5] \cup (2.75, 3] \cup (3.25, 3.5]$. Using the upper bound of Lemma 5 (which holds for FF) we show the following:

Theorem 6. *The approximation ratios of* FF *in the following intervals:* $\alpha \in (1, 1.5]$, $\alpha \in (1.5, 2]$, $\alpha \in (2, 2.5]$, $\alpha \in (2.75, 3]$, *and* $\alpha \in (3.25, 3.5]$ *are* $1, \frac{3}{2}, \frac{4}{3}, \frac{5}{3}$ *and* $\frac{3}{2}$, *respectively.*

Proof. If $\alpha \in (1, 1.5]$, then by Corollary 3, $1 < \text{FF} \le 2$, Clearly, $\text{OPT} \ge 2$.

If $\alpha \in (1.5, 2]$, then by Corollary 3, $\text{FF} \le 3$ and $\text{OPT} \ge 2$. The following example shows that this bound is tight: Let $\varepsilon_1 = \frac{\alpha-1.5}{3} \le \frac{1}{6}$. There are four items, of sizes $\frac{1}{2} - \varepsilon_1, 2\varepsilon_1, \frac{1}{2} + \varepsilon_1, \frac{1}{2} + \varepsilon_1$, in that order. The total size of the items is $\frac{3}{2} + 3\varepsilon_1 = \alpha$. FF will use three bins with a total size of items of $\frac{1}{2} + \varepsilon_1$ in each bin. An optimal solution uses one bin for the first item and the last item, whose total size is 1, and the remaining items, whose total size is at most 1 are packed in an additional bin.

If $\alpha \in (2, 2.5]$, then $\text{FF} \le 4$ and $\text{OPT} \ge 3$. The following example shows that this bound is tight: Let $\varepsilon_2 = \frac{\alpha-2}{4} \le \frac{1}{8}$. There are five items, of sizes $\frac{1}{2} - \varepsilon_2, 2\varepsilon_2, \frac{1}{2} + \varepsilon_2, \frac{1}{2} + \varepsilon_2, \frac{1}{2} + \varepsilon_2$. The total size of the items is $2 + 4\varepsilon_2 = \alpha$. FF will use four bins with a total size of items of $\frac{1}{2} + \varepsilon_2$ in each bin. An optimal solution uses one bin for the first item and the last item, whose total size is 1, one bin for another item of size $\frac{1}{2} + \varepsilon_2$, and one final bin for the remaining items, whose total size is less than 1.

If $\alpha \in (2.75, 3]$, then $\text{FF} \le 5$ and $\text{OPT} \ge 3$. The following example shows that this bound is tight: Let $\varepsilon_3 = \frac{\alpha-2.75}{4} \le \frac{1}{16}$. There are nine items; three items of the size $\frac{1}{4} - \varepsilon_3$, one item of size $4\varepsilon_3$, two items of size $\frac{1}{4}$, and three items of size $\frac{1}{2} + \varepsilon_3$. The total size of the items is $3(\frac{1}{4} - \varepsilon_3) + 4\varepsilon_3 + 2 \cdot \frac{1}{4} + 3(\frac{1}{2} + \varepsilon_3) = 2.75 + 4\varepsilon_3 = \alpha$. FF will use five bins with total sizes of items $\frac{3}{4} + \varepsilon_3, \frac{1}{2}, \frac{1}{2} + \varepsilon_3, \frac{1}{2} + \varepsilon_3, \frac{1}{2} + \varepsilon_3$. An optimal solution has two bins with items of sizes $\frac{1}{4} - \varepsilon_3$, $\frac{1}{4}$ and $\frac{1}{2} + \varepsilon_3$, and one bin with the remaining items, whose total size is at most 1.

If $\alpha \in (3.25, 3.5]$, then $\text{FF} \le 6$ and $\text{OPT} \ge 4$. The following example shows that this bound is tight: Let $\varepsilon_4 = \frac{\alpha-3.25}{5} \le \frac{1}{20}$. There are ten items; three items of the size $\frac{1}{4} - \varepsilon_4$, one item of size $4\varepsilon_4$, two items of size $\frac{1}{4}$, and four items of size $\frac{1}{2} + \varepsilon_4$. The total size of the items is $3(\frac{1}{4} - \varepsilon_4) + 4\varepsilon_4 + 2 \cdot \frac{1}{4} + 4(\frac{1}{2} + \varepsilon_4) = 3.25 + 5\varepsilon_4 = \alpha$. FF will use six bins, one with total size of items $\frac{3}{4} + \varepsilon_4$, one with $\frac{1}{2}$, and four with $\frac{1}{2} + \varepsilon_4$ each. An optimal solution has two bins with items of sizes $\frac{1}{4} - \varepsilon_4$, $\frac{1}{4}$ and $\frac{1}{2} + \varepsilon_4$, a bin with a single item of size $\frac{1}{2} + \varepsilon_4$, and one bin with the remaining items, whose total size is less than 1. $\square$

We next prove properties which are useful also later in the paper. We use the following notation: Let a t-bin denote a bin, packed by FF, which contains exactly t items, and a t^+-bin, a bin which contains *at least* t items.

Remark 7. It can be assumed that no bin of FF contains two items which are packed in the same bin by OPT.

Proof. If two such items exist, combine them into one item which arrives at the time when the first of these two items occurs in the sequence. The packing of FF does not change since every item can still be packed to the same bin as before, and no item can fit into an earlier bin as a result of the merge. $\square$

Lemma 8. *Suppose $p > k \geq 1$ are integers, and we are given a packing of* FF *in which there are p bins, each containing at least k items. Then the total size of the items in these bins is more than $\frac{pk}{k+1}$, i.e., the average load per bin exceeds $\frac{k}{k+1}$.*

Proof. Note first that at most one of our p bins can have total contents less than or equal to $\frac{k}{k+1}$, for if there were two, then the second would have to contain k items of size greater than $\frac{1}{k+1}$, for a total exceeding $\frac{k}{k+1}$, which is a contradiction. Consider the last $p-k \geq 1$ of our p bins, and pick one of them in the following way. If one of these bins has a load less than or equal to $\frac{k}{k+1}$, pick that, and otherwise pick any one. Take the first k items from the selected bin, and assign one each to the first k of our p bins. By FF, none of these items will fit, so the total size of each of these first k bins, plus their assigned items, will be greater than k. Since all unpicked bins among the last $p-k$ have a load of at least $\frac{k}{k+1}$, this yields a total size exceeding $k + (p-k-1)(\frac{k}{k+1}) = p(\frac{k}{k+1}) + k - (k+1)(\frac{k}{k+1}) = \frac{pk}{k+1}$. □

We let N_1 denote the number of 1-bins in a given packing created by FF. We call the items packed into these bins *big* items. Clearly, no two such items fit into a bin together, as otherwise FF would have combined them into one bin. Thus the following holds.

Lemma 9. *All big items (if at least one such bin exists) come from distinct bins in the optimal packing, thus $N_1 \leq$ OPT.*

Lemma 10. *Let OPT $= \lambda$. If* FF *creates λ 1-bins, then all items in the 2-bins (if such bins exist) come from distinct bins in the optimal packing, i.e., there are at most $\lfloor \frac{\lambda}{2} \rfloor$ 2-bins.*

Proof. By Lemma 9, since there are λ 1-bins in FF's packing, every bin of the optimal packing contains a big item. Assume for the sake of contradiction that a bin b in OPT's packing contains items x and y, both of which were in 2-bins in FF's packing. By Remark 7, x and y are not in the same 2-bin in the packing of FF. Without loss of generality, assume that FF opened the bin where it put x before opening the bin with y.

Let Z be the big item which OPT packs with x and y. Let a be the item FF packed in a 2-bin with x. Let C be the big item which OPTpacked with a. (We let x, y, Z, a, C denote both the items and their sizes.)

We have $Z + x + y \leq 1$. The item y does not fit into the bin of x in the execution of FF. Regardless of whether or not a is already packed at that time, we have $a + x + y > 1$. Thus, $a > Z$. However, C and Z are packed into 1-bins by FF, so $C + Z > 1$. Since $a + C \leq 1$, we get $Z + C < a + C \leq 1$, which is a contradiction. □

Lemma 11. *If* FF $\geq$ OPT $+ 2$ *and* FF ≥ 4, *then* $\alpha > \frac{2 \cdot \text{FF}+1}{4}$.

Proof. Since $N_1 \leq$ OPT, the number of 2^+-bins is FF $- N_1 \geq$ FF $-$ OPT ≥ 2. We consider two cases based on the total size of the items which FF packs in the first 2^+-bin. If that total size is at least $\frac{3}{4}$, then applying Corollary 3, the FF $- 1 \geq 3 > 2$ other bins contain items with total size more than $\frac{\text{FF}-1}{2}$. Thus, $\alpha > \frac{3}{4} + \frac{\text{FF}}{2} - \frac{1}{2} = \frac{\text{FF}}{2} + \frac{1}{4}$.

In the second case, let $\frac{3}{4} - \delta$ for some $\delta > 0$ denote the total size of the items in the first 2^+-bin. The second 2^+-bin has at least two items, each of which is larger than $\frac{1}{4} + \delta$. The total size of items in these two bins is thus more than $\frac{3}{4} - \delta + 2(\frac{1}{4} + \delta) = \frac{5}{4} + \delta > \frac{5}{4}$. Again applying Corollary 3 with $k = 1$, the FF $- 2 \geq 2$ other bins have a total size of more than $\frac{\text{FF}-2}{2}$, so $\alpha > \frac{5}{4} + \frac{\text{FF}}{2} - 1 = \frac{\text{FF}}{2} + \frac{1}{4}$. □

Corollary 12. *The approximation ratio of* FF *for $\alpha \in (2.5, 2.75]$ is at most $\frac{4}{3}$. The approximation ratio of* FF *for $\alpha \in (3, 3.25]$ is at most $\frac{5}{4}$.*

Proof. In the first case, OPT ≥ 3, and FF ≤ 5 (by Lemma 5). The only potential case where the approximation ratio is larger than $\frac{4}{3}$ is OPT $= 3$ and FF $= 5$. By Lemma 11, in this case $\alpha > \frac{11}{4} = 2.75$, which is a contradiction.

In the second case, OPT ≥ 4, and FF ≤ 6 (by Lemma 5). The only potential case where the approximation ratio is larger than $\frac{5}{4}$ is OPT $= 4$ and FF $= 6$. By Lemma 11, in this case $\alpha > \frac{13}{4} = 3.25$, which is a contradiction. □

The following lemma, a slight strengthening of the bound of Simchi-Levi [9] (who showed FF ≤ 1.75OPT), will be helpful for handling the situation when $\alpha > 3.5$. We prove an additional useful property in the same lemma.

Lemma 13. *Assume $\frac{\text{FF}}{\text{OPT}} > \frac{12}{7}$ and* OPT ≥ 3. *Then the following properties hold:*

- *There are at least three 1-bins, that is, $N_1 \geq 3$.*
- $\frac{\text{FF}}{\text{OPT}} < 1.75$.

Proof. We have FF $> \frac{12}{7}$OPT > 5, so FF ≥ 6.

Suppose first that $N_1 \leq 2$.

Using Lemma 8 for the 2^+-bins, we get that the sum of the sizes satisfies

$$\alpha > \frac{2}{3}(\text{FF} - 2) + 1 > \frac{2}{3} \cdot \frac{12}{7}\text{OPT} - \frac{1}{3} > \text{OPT},$$

since $\frac{\text{OPT}}{7} > \frac{1}{3}$, which is contradiction. Thus $N_1 \geq 3$.

Since $N_1 \leq \textsc{Opt}$, the number of 2^+-bins is larger than $\frac{5}{7}\textsc{Opt} > 2$, i.e., there are at least three 2^+-bins, so Lemma 8 can be used for these bins.

If $\text{FF}/\textsc{Opt} \geq 7/4$ holds, using $2 < N_1 \leq \textsc{Opt}$, and applying Corollary 3 on the 1-bins, we get

$$\alpha > \frac{2}{3}\left(\frac{7}{4}\textsc{Opt} - N_1\right) + \frac{1}{2}N_1 = \frac{7}{6}\textsc{Opt} - \frac{1}{6}N_1 \geq \textsc{Opt}$$

which is a contradiction. Thus $\text{FF}/\textsc{Opt} < 7/4$ follows. □

Lemma 14. *If* $\text{FF} \geq 2\textsc{Opt} - 2$ *and* $\textsc{Opt} \geq 5$, *then* $\alpha > \textsc{Opt} - \frac{1}{2}$.

Proof. Since $N_1 \leq \textsc{Opt}$, and $\text{FF} - N_1 \geq 2\textsc{Opt} - 2 - N_1 \geq \textsc{Opt} - 2 \geq 3$, we apply Lemma 8 to the 2^+-bins and Corollary 3 to the 1-bins, to get $\alpha > \frac{2}{3}(\text{FF} - N_1) + \frac{N_1}{2} = \frac{2}{3}\text{FF} - \frac{N_1}{6} \geq \frac{2}{3}(2\textsc{Opt} - 2) - \frac{\textsc{Opt}}{6} = \frac{7}{6}\textsc{Opt} - \frac{4}{3} \geq \textsc{Opt} - \frac{1}{2}$, for $\textsc{Opt} \geq 5$. □

The comparison between FF and WF for $\alpha > 6.5$ will follow from, the upper bound of $\frac{12}{7}$ on the approximation ratio of FF, proved in the next section (Theorem 26), which is lower than the approximation ratio of WF, which was shown to be at least $\frac{7}{4}$ in this case (in Lemma 5—this $\frac{7}{4}$ bound is exact for $\alpha \in (7, 7.5]$). We next consider the remaining cases.

Theorem 15. *For* $3.5 < \alpha \leq 6.5$, $\mathcal{R}_{\text{FF}}(\alpha)$ *obeys the upper bounds specified in Table 1. All of these bounds are smaller than the corresponding values for* $\mathcal{R}_{\text{WF}}(\alpha)$ *from Lemma 5, also included in the table.*

Proof. In the cases where $\alpha \in (a, a + \frac{1}{2}]$ for $a = 4, 5, 6$, $\textsc{Opt} \geq a + 1$. Using Corollary 3 and the fact that FF is an integer, $\text{FF} < 2\alpha \leq 2a + 1$, so $\text{FF} \leq 2a$. If $\textsc{Opt} = a + 1$, and $\text{FF} = 2a$, then by Lemma 14, $\alpha > a + \frac{1}{2}$, so this case is impossible. The approximation ratio is therefore at most $\max\{\frac{2a}{a+2}, \frac{2a-1}{a+1}\} = \frac{2a-1}{a+1}$, which is equal to $\frac{7}{5}, \frac{3}{2}, \frac{11}{7}$, in the intervals $(4, 4.5]$, $(5, 5.5]$, and $(6, 6.5]$, respectively.

In the case $\alpha \in (3.5, 4]$, $\textsc{Opt} \geq 4$. If $\textsc{Opt} = 4$, then by Lemma 13, $\text{FF} < 1.75\textsc{Opt} = 7$, so $\text{FF} \leq 6$. If $\textsc{Opt} \geq 5$, Lemma 5 implies that $\text{FF} \leq 7$. Therefore, the approximation ratio is at most 1.5.

In the case $\alpha \in (4.5, 5]$, $\textsc{Opt} \geq 5$. If $\textsc{Opt} = 5$, then by Lemma 13, $\text{FF} < 1.75\textsc{Opt} = 8.75$, so $\text{FF} \leq 8$. If $\textsc{Opt} \geq 6$, Lemma 5 implies that $\text{FF} \leq 9$. Therefore, the approximation ratio is at most 1.6.

In the case $\alpha \in (5.5, 6]$, $\textsc{Opt} \geq 6$. If $\textsc{Opt} = 6$, then by Lemma 13, $\text{FF} < 1.75\textsc{Opt} = 10.5$, so $\text{FF} \leq 10$. If $\textsc{Opt} \geq 7$, Lemma 5 implies that $\text{FF} \leq 11$. Therefore, the approximation ratio is at most $\frac{5}{3}$. □

4. An absolute upper bound on the approximation ratio of FF

The following is an example showing that 1.7 is a lower bound on FF's absolute approximation ratio; another example can be found in [8]:

Let ϵ be sufficiently small. The input sequence consist of ten items of size $\frac{1}{30} - 2\epsilon$, ten of size $\frac{2}{15}$, ten of size $\frac{1}{3} + \epsilon$, and ten of size $\frac{1}{2} + \epsilon$. A bin could contain one item of each size, so $\textsc{Opt} = 10$. If the items are given in increasing order, FF places all of the smallest items and five of those of size $\frac{2}{15}$ in the first bin. The remaining items of size $\frac{2}{15}$ are placed in the second bin. The next five bins each contain two items of size $\frac{1}{3} + \epsilon$, and the last ten bins each contain one item of size $\frac{1}{2} + \epsilon$. Thus, FF uses 17 bins.

In this section we prove a new upper bound, $\frac{12}{7}$, on the absolute approximation ratio for FF. In the previous section, we showed upper bounds on this approximation ratio for the cases where $\alpha \in (0, 6.5]$, all of which were strictly below $\frac{12}{7} \approx 1.71429$. Thus, in order to prove the result for all α, we now consider $\alpha > 6.5$, and assume that $\textsc{Opt} \geq 7$ throughout this section. All sequences with $\textsc{Opt} \geq 7$ will be considered, and thus all values $\alpha > 6.5$ will be considered, but α will not be used explicitly.

We use the previously known upper bound $\text{FF} \leq \lceil 1.7\textsc{Opt} \rceil$, proved by Garey et al. [6], and the second part of Lemma 13, that is, that $\text{FF} < 1.75\textsc{Opt}$. We find the following:

Lemma 16. *For the following values of* Opt, *the maximum possible value of* FF *does not exceed* $\frac{12}{7}\textsc{Opt}$.

1. *All values of* Opt *exceeding* 70.
2. *All values of* Opt *divisible by* 10.
3. *All values of* Opt *divisible by* $10q + 7$ *for nonnegative integers* q *(values of* Opt *divisible by* 7, 17, 27, 37, … *).*
4. *The values* 8, 9, 12, 13, 16.
5. *The values* 24, 31, 38, 41, 44, 45, 48, 52, 55, 58, 59, 61, 62, 64, 65, 66, 69.

The values of Opt for which the claimed bounds do not imply $\text{FF} \leq \frac{12}{7}\textsc{Opt}$ are 11, 15, 18, 19, 22, 23, 25, 26, 29, 32, 33, 36, 39, 43, 46, and 53, and we deal with these values later.

Proof. If $\textsc{Opt} \geq 71$, then $\lceil 1.7\textsc{Opt} \rceil < 1.7\textsc{Opt} + 1 \leq 1.7\textsc{Opt} + \frac{\textsc{Opt}}{71} < \frac{12}{7}\textsc{Opt}$.

If $\textsc{Opt} = 10q$, for an integer q, where $q \geq 1$, then $\lceil 1.7\textsc{Opt} \rceil = \lceil 17q \rceil = 17q = 1.7\textsc{Opt}$.

If $\textsc{Opt} = (10q + 7)i$, for integers q, i, where $q \geq 0$ and $i \geq 1$, then $\lceil 1.7\textsc{Opt} \rceil = \lceil 1.7 \cdot (10q + 7)i \rceil = \lceil 17qi + 11.9i \rceil \leq 17qi + 12i \leq \frac{12}{7}(10qi + 7i) = \frac{12}{7}\textsc{Opt}$. Note that if $q > 0$, then a slightly smaller upper bound can be shown (since in this case $\frac{12}{7} \cdot 10qi > 17qi$).

The remaining cases are given in the following tables:

Opt	8	9	12	13	16
$\lceil 1.75 \cdot \text{Opt} \rceil - 1$	13	15	20	22	27
Approximation ratio	1.625	<1.67	<1.67	<1.7	<1.7

Opt	24	31	38	41	44	45	48	52	55
$\lceil 1.7\text{Opt} \rceil$	41	53	65	70	75	77	82	89	94
Approximation ratio	<1.71	<1.71	<1.7106	<1.71	<1.71	<1.712	<1.71	<1.712	<1.71

Opt	58	59	61	62	64	65	66	69
$\lceil 1.7\text{opt} \rceil$	99	101	104	106	109	111	113	118
Approximation ratio	<1.71	<1.712	<1.71	<1.71	<1.71	<1.71	<1.713	<1.711

□

The remaining values of Opt to be considered are all larger than 10. Therefore we assume $\text{Opt} \geq 11$ in what follows, and if the approximation ratio is violated, then $\text{FF} - \text{Opt} > \frac{5\text{Opt}}{7} > 7$, that is, $\text{FF} - \text{Opt} \geq 8$. Assume for the sake of contradiction that there exists a counterexample to the claim $\mathcal{R}_{\text{FF}} \leq \frac{12}{7}$.

We define a class of items, called *bad items*. These are items packed by FF into 2^+-bins, which have a size strictly larger than $\frac{1}{4}$. We use B to denote the number of bad items.

Lemma 17. *If each big item is larger than* $1/2$*, then* $B \leq 3\text{Opt} - 2N_1$*. If there is a big item of a size in* $(0, \frac{1}{4}]$*, then* $B \leq 3\text{Opt} - 2N_1$*. If there is a big item of a size in* $(\frac{1}{4}, \frac{1}{2}]$*, then* $B \leq 3\text{Opt} - 2N_1 + 1$*. In the last two cases, all* 2^+*-bins contain a total size of items which exceeds* $\frac{1}{2}$*.*

Proof. There are at least $N_1 - 1$ big items larger than $1/2$, since the total size packed into a pair of such bins exceeds 1. Every bin of Opt which does not contain a big item can contain at most three bad items. A bin of Opt containing a big item of size larger than $\frac{1}{2}$, can contain at most one bad item. Therefore, if all big items are larger than $\frac{1}{2}$, the number of bad items is at most $3(\text{Opt} - N_1) + N_1 = 3\text{Opt} - 2N_1$.

Otherwise, there is a big item of a size $z \in (0, \frac{1}{2}]$. If $z > \frac{1}{4}$, then its bin in Opt can contain at most two bad items, so the number of bad items is at most $3\text{Opt} - 2N_1 + 1$. Otherwise, all other big items have a size larger than $\frac{3}{4}$, so their bins in Opt do not have any bad items at all. Therefore $B \leq 3(\text{Opt} - N_1) + 3 \leq 3\text{Opt} - 2N_1$, since $N_1 \geq 3$ by Lemma 13.

The last claim holds since the total size of every pair of bins exceeds 1, and there is a 1-bin with a total size no larger than $\frac{1}{2}$. □

Lemma 18. *Without loss of generality, one can assume that the* 1*-bins are the last* N_1 *bins in* FF*'s packing.*

Proof. Moving the items packed into these bins to the end of the sequence does not change the packing of the other bins. These N_1 items cannot be packed into fewer than N_1 bins, and cannot join previous bins. □

We let J denote the maximum length prefix of bins packed by FF which each have a total size of items larger than $\frac{3}{4}$.

Lemma 19. $J \geq \text{FF} - \frac{3}{2}\text{Opt} - \frac{3}{2}$.

Proof. Assume for the sake of contradiction that $J \leq \text{FF} - \frac{3}{2}\text{Opt} - 2$. In this case $J + 2 \leq \text{FF} - \frac{3}{2}\text{Opt} < \text{FF} - \text{Opt} \leq \text{FF} - N_1$, therefore $J + 1 < \text{FF} - N_1$. Consider the first $\text{FF} - N_1$ bins, all of which must be 2^+-bins, by Lemma 18. There are at least

$$\text{FF} - N_1 - (J + 1) \geq \text{FF} - N_1 - \left(\text{FF} - \frac{3}{2}\text{Opt} - 1\right) = \frac{3\text{Opt}}{2} - N_1 + 1$$

2^+-bins after the first $J + 1$ bins. The $(J + 1)$-st bin has a total size of at most $\frac{3}{4}$, therefore all items in the subsequent $\frac{3\text{Opt}}{2} - N_1 + 1 > 0$ 2^+-bins are bad. Since there are at least two items in each of these bins, $B \geq 3\text{Opt} - 2N_1 + 2$, contradicting Lemma 17. □

Lemma 20. $\text{FF} - N_1 - J \geq 3$.

Proof. Suppose that $\text{FF} - N_1 - J \leq 2$, and thus $J \geq \text{FF} - N_1 - 2 \geq \text{FF} - \text{Opt} - 2$.

Assuming that $\text{FF} - J \geq 2$, one can apply Corollary 3 to the last $\text{FF} - J \geq 2$ bins and get:

$$\text{Opt} \geq \alpha > \tfrac{3}{4}J + \tfrac{\text{FF}-J}{2} = \tfrac{\text{FF}}{2} + \tfrac{J}{4} \geq \tfrac{\text{FF}}{2} + \tfrac{\text{FF}-\text{Opt}-2}{4} = \tfrac{3}{4}\text{FF} - \tfrac{1}{4}\text{Opt} - \tfrac{1}{2}.$$

That is, $\frac{3}{4}\text{FF} < \frac{5}{4}\text{Opt} + \frac{1}{2}$, which gives $\text{FF} \leq \frac{5}{3}\text{Opt} + \frac{1}{3}$. Using $\text{Opt} \geq 11$, this gives $\text{FF} \leq \frac{5}{3}\text{Opt} + \frac{\text{Opt}}{33} < 1.7\text{Opt}$, which is a contradiction.

On the other hand, if $\mathrm{FF} - J \le 1$, then, since $\alpha > \frac{3}{4}J$ and $\textsc{Opt} \ge 11$, we have that

$$\mathrm{FF} \le J + 1 < \frac{4}{3}\alpha + 1 \le \frac{4}{3}\textsc{Opt} + 1 \le \frac{47}{33}\textsc{Opt} < 1.7\textsc{Opt},$$

which is also a contradiction. □

Lemma 21. $J < 12\textsc{Opt} - 8\mathrm{FF} + 2N_1$.

Proof. Assume for the sake of contradiction that $J \ge 12\textsc{Opt} - 8\mathrm{FF} + 2N_1$. Since $\mathrm{FF} - N_1 - J \ge 3$, we consider the first J bins separately from the other bins, and apply Lemma 8 on the next $\mathrm{FF} - J - N_1 \ge 3$ bins, which are 2^+-bins, and on the last N_1 bins, giving:

$$\begin{aligned}\textsc{Opt} \ge \alpha &> \frac{3}{4}J + \frac{2}{3}(\mathrm{FF} - J - N_1) + \frac{N_1}{2} \ge \frac{2\mathrm{FF}}{3} + \frac{J}{12} - \frac{N_1}{6}\\ &\ge \frac{2\mathrm{FF}}{3} + \frac{12\textsc{Opt} - 8\mathrm{FF} + 2N_1}{12} - \frac{N_1}{6} = \textsc{Opt},\end{aligned}$$

which is a contradiction. □

Lemma 22. 1. *If* $N_1 < \textsc{Opt}$, *then* $\mathrm{FF} \le \frac{31}{18}\textsc{Opt} - \frac{1}{6}$.
2. *If* $\textsc{Opt} \ge 15$, *then the following properties hold:*
- *if* $\textsc{Opt}$ *is even, then* $\mathrm{FF} \le \frac{31}{18}\textsc{Opt} - \frac{1}{18}$
- *if* $\textsc{Opt}$ *is odd, then* $\mathrm{FF} \le \frac{31}{18}\textsc{Opt} - \frac{1}{9}$.

Proof. Using the last three lemmas gives $\mathrm{FF} - \frac{3}{2}\textsc{Opt} - \frac{3}{2} \le J \le 12\textsc{Opt} - 8\mathrm{FF} + 2N_1 - 1$. If $N_1 \le \textsc{Opt} - 1$, we get $\mathrm{FF} - \frac{3}{2}\textsc{Opt} - \frac{3}{2} \le 12\textsc{Opt} - 8\mathrm{FF} + 2\textsc{Opt} - 3$, or $18\mathrm{FF} \le 31\textsc{Opt} - 3$.

The second part follows from the first part if $N_1 < \textsc{Opt}$. If $N_1 = \textsc{Opt}$, by Lemma 10, there are at most $\lfloor \frac{\textsc{Opt}}{2} \rfloor$ 2-bins. If $\mathrm{FF} \le \frac{31}{18}\textsc{Opt} - \frac{1}{9}$, then the claimed bounds for both even and odd values of $\textsc{Opt} = N_1$ would hold, so assume instead that $\mathrm{FF} > \frac{31}{18}\textsc{Opt} - \frac{1}{9}$. We get that there are at least $\mathrm{FF} - \frac{3\textsc{Opt}}{2} > \frac{2}{9}\textsc{Opt} - \frac{1}{9}$ 3^+-bins. If $\textsc{Opt} \ge 15$, then $\frac{2\textsc{Opt}}{9} - \frac{1}{9} > 3$, and thus Lemma 8 can be applied on the 3^+-bins. We apply the lemma on the $N_1 = \textsc{Opt}$ 1-bins, on $\lfloor \frac{\textsc{Opt}}{2} \rfloor \ge 7$ bins containing all 2-bins, and on the remaining (at least 4) bins, which are all 3^+-bins.

Therefore $\textsc{Opt} \ge \alpha > \frac{\textsc{Opt}}{2} + \frac{2}{3}(\lfloor \frac{\textsc{Opt}}{2} \rfloor) + \frac{3}{4}(\mathrm{FF} - \textsc{Opt} - \lfloor \frac{\textsc{Opt}}{2} \rfloor) = \frac{3}{4}\mathrm{FF} - \frac{\textsc{Opt}}{4} - \frac{\lfloor \frac{\textsc{Opt}}{2} \rfloor}{12} \ge \frac{3}{4}\mathrm{FF} - \frac{7}{24}\textsc{Opt}$. This gives $\mathrm{FF} < \frac{31}{18} \cdot \textsc{Opt}$ or $\mathrm{FF} \le \frac{31}{18}\textsc{Opt} - \frac{1}{18}$. If $\textsc{Opt}$ is odd, then $\lfloor \frac{\textsc{Opt}}{2} \rfloor = \frac{\textsc{Opt}-1}{2}$. Then $\textsc{Opt} > \frac{3}{4}\mathrm{FF} - \frac{\textsc{Opt}}{4} - \frac{\lfloor \frac{\textsc{Opt}}{2} \rfloor}{12} = \frac{3}{4}\mathrm{FF} - \frac{7}{24}\textsc{Opt} + \frac{1}{24}$, which is equivalent to $31\textsc{Opt} - 1 > 18\mathrm{FF}$ or $31\textsc{Opt} - 2 \ge 18\mathrm{FF}$. That is, $\mathrm{FF} \le \frac{31}{18}\textsc{Opt} - \frac{1}{9}$. □

Before we proceed, we consider the case $\textsc{Opt} = 11$, which is not covered by the previous lemma.

Lemma 23. *If* $\textsc{Opt} = 11$, *then* $\mathrm{FF} \le 18 < \frac{12}{7}\textsc{Opt}$.

Proof. The case $N_1 \le 10$ follows from Lemma 22, part 1. Assume, therefore, that $\mathrm{FF} \ge 19$ and $N_1 = 11$. From the bounds on J we get $J < 132 - 8FF + 22$. Since $J \ge 0$, $\mathrm{FF} = 19$ and $J \le 1$. By Lemma 10, there can be at most five 2-bins among the bins of indices $3, 4, \ldots, 8$. Therefore, there exists at least one 3-bin among these bins. This allows to tighten the bound on the number of bad items as follows. Since at least one of the first two bins has a total size of at most $\frac{3}{4}$, there are at least 13 bad items in bins $3, 4, \ldots, 8$, contradicting Lemma 17. □

Lemma 24. *For the following values of* $\textsc{Opt}$, *the maximum possible value of* FF *does not exceed* $\frac{12}{7}\textsc{Opt}$: 15, 18, 19, 22, 23, 25, 26, 29, 33, 36, 43.

Proof. We use the second part of Lemma 22. The resulting upper bounds on FF and the approximation ratios are as follows (the first table is for the odd values of $\textsc{Opt}$ and the second table is for the even values).

$\textsc{Opt}$	15	19	23	25	29	33	43
$\lfloor \frac{31}{18} \cdot \textsc{Opt} - 1/9 \rfloor$	25	32	39	42	49	56	73
Approximation ratio	5/3	<1.69	<1.70	1.68	<1.69	<1.70	<1.70

$\textsc{Opt}$	18	22	26	36
$\lfloor \frac{31}{18} \cdot \textsc{Opt} - 1/18 \rfloor$	30	37	44	61
Approximation ratio	5/3	<1.69	<1.70	<1.70

□

The remaining values of Opt to be considered are only 32, 39, 46, and 53.

Lemma 25. *For the cases* $\text{Opt} = 32$, $\text{Opt} = 39$, $\text{Opt} = 46$, $\text{Opt} = 53$, *the costs of* FF *are at most* 54, 66, 78 *and* 90, *respectively. Thus the approximation ratios is these cases are below* 1.7.

Proof. Since $\text{FF} \leq \lceil 1.7\text{Opt} \rceil$, we need to exclude the cases $\text{Opt} = 32$ and $\text{FF} = 55$, $\text{Opt} = 39$ and $\text{FF} = 67$, $\text{Opt} = 46$ and $\text{FF} = 79$, and finally, $\text{Opt} = 53$ and $\text{FF} = 91$.

Using Lemmas 19 and 21, we find the following bounds on J in these four cases. Let $k' = \text{Opt} - N_1$.

In the first case $6 \leq J \leq 7 - 2k'$. Therefore, $k' = 0$, $N_1 = \text{Opt} = 32$ and $J \in \{6, 7\}$.

In the second case $7 \leq J \leq 9 - 2k'$. Therefore there are two cases for N_1. If $N_1 = \text{Opt} = 39$, then $J \in \{7, 8, 9\}$. If $N_1 = \text{Opt} - 1 = 38$, then $J = 7$.

In the third case $9 \leq J \leq 11 - 2k'$. Therefore there are two cases for N_1. If $N_1 = \text{Opt} = 46$, then $J \in \{9, 10, 11\}$. If $N_1 = \text{Opt} - 1 = 45$, then $J = 9$.

In the fourth case $10 \leq J \leq 13 - 2k'$. Therefore there are two cases for N_1. If $N_1 = \text{Opt} = 53$, then $J \in \{10, 11, 12, 13\}$. If $N_1 = \text{Opt} - 1 = 52$, then $J \in \{10, 11\}$.

We thus have 16 cases. Note that in all cases, we have $\text{FF} > J + N_1 + 1$. The remainder of the proof has the following structure for each of the above cases:

Let B' be the number of bad items (ones of size exceeding $\frac{1}{4}$) in the first J bins, and let r be the number of bins among the first J that do not contain any bad items. Note that $r \geq J - B'$. We use our earlier lemmas to obtain bounds on B' and hence r, and then to prove a lower bound on the total size of items in terms of r, J, N_1, Opt, and FF that applies in all but one case (which we handle separately). For each of the other cases, we show that for the specified values of J, N_1, Opt, and FF, the lower bound on total size exceeds Opt, thus yielding the desired contradiction.

The sequence of bins which are located after the first $J + 1$ bins and before the last N_1 bins is studied in detail. We find a lower bound on the number of bad items in those bins. Given the upper bound on B from Lemma 17, we find an upper bound on B'. Specifically, each bin in J has items with total size more than $\frac{3}{4}$, so each one of the r bins with no bad items must contain at least four items. If there are at least five such bins with at least four items, they can be analyzed separately, and they have an average total size which exceeds $\frac{4}{5}$, by Lemma 8. Moreover, since the $J + 1$-st bin has a total size of at most $\frac{3}{4}$, all subsequent 2^+-bins have at least two bad items. The number of such bins is $\text{FF} - N_1 - J - 1 > 0$.

Using the bound on B from Lemma 17 ($B \leq 3\text{Opt} - 2N_1 + 1$), and the number of 2^+-bins that must contain only bad items, we find an upper bound on B' and hence a lower bound on r. In all cases except for the one with $\text{Opt} = 32$, $\text{FF} = 55$, $N_1 = 32$, and $J = 7$, which will be treated separately, we are guaranteed that $r \geq 5$ and that there are at least three 2+-bins after the first J bins (by Lemma 20), and hence Lemma 8 can be exploited for both groups of bins, implying that the total item size is at least

$$\frac{3}{4}(J - r) + \frac{4}{5}r + \frac{2}{3}(\text{FF} - J - N_1) + \frac{N_1}{2} = \frac{r}{20} + \frac{J}{12} + \frac{2\text{FF}}{3} - \frac{N_1}{6}. \tag{1}$$

Table 2 summarizes all cases which need to be considered, and lists the bounds on B, B', r, and the total size derived for each of these cases, indicating the desired contradiction in all but the first case. As an example, consider the second row. We have $\text{FF} - N_1 - J - 1 = 16$, so at least 32 bad items are packed outside of the first J bins. By the upper bound on B, there is at most one bad item in the first 6 bins. Thus, there are five bins among the first $J = 6$ bins containing at least four items each, and thus an average total size of at least $\frac{4}{5}$. An additional bin, out of the first 6 bins, which is not analyzed with the other five bins, has a total size of at least $\frac{3}{4}$. The next 17 bins are 2^+-bins which have an average total size of at least $\frac{2}{3}$. The last $N_1 = 32$ bins have an average total size of at least $\frac{1}{2}$. The total size is at least $\frac{385}{12} > 32$, which is a contradiction.

We are left with the first case with $r = 4$. Consider four bins which are 4^+-bins. Remove the smallest three items from the last such bin, and add each one of them to a previous bin in this set. This gives total size which exceeds 3 for these bins. The remainder in the fourth bin is at least a quarter of its total size, i.e., at least $\frac{3}{16}$. Thus the total size of items exceeds $3 + \frac{3}{16} + \frac{3}{4}(J - r) + \frac{2}{3}(\text{FF} - J - N_1) + \frac{N_1}{2} = 3 + \frac{3}{16} - \frac{3}{4}r + \frac{J}{12} + \frac{2\text{FF}}{3} - \frac{N_1}{6} > 32.104 > \text{Opt}$, which gives a contradiction in this case as well. $\square$

In Section 3, the absolute approximation ratio of FF was shown to be strictly less than $\frac{12}{7}$ for all cases where $\alpha \leq 6.5$. In this section, the absolute approximation ratio of FF was shown to be at most $\frac{12}{7}$ whenever $\text{Opt} \geq 7$. This covers all cases, giving us the following:

Theorem 26. *The absolute approximation ratio of* FF *is at most* $\frac{12}{7}$.

5. Conclusion and open problems

The comparison between WF and NF showed that for all values of α, the sum of the sizes of all input elements, WF is at least as good as NF. This follows immediately from the result that WF performs at least as well as NF on every input sequence. In addition, for some values of α, including all values of α greater than 6, WF has a better approximation ratio than NF.

The comparison between FF and WF showed that for all values of α FF is at least as good as WF. FF has a better approximation ratio than WF for many ranges of α, including all α greater than 3.5.

Table 2
The 16 cases which cover the remaining values of Opt: 32, 39, 46, 53.

Opt	FF	N_1	J	UB on B	UB on B'	LB on $r = J - B'$	LB on total size
32	55	32	7	33	3	4	
32	55	32	6	33	1	5	>32
39	67	39	9	40	4	5	>39
39	67	39	8	40	2	6	>39
39	67	39	7	40	0	7	>39
39	67	38	7	42	0	7	>39
46	79	46	11	47	5	6	>46
46	79	46	10	47	3	7	>46
46	79	46	9	47	1	8	>46
46	79	45	9	49	1	8	>46
53	91	53	13	54	6	7	>53
53	91	53	12	54	4	8	>53
53	91	53	11	54	2	9	>53
53	91	53	10	54	0	10	>53
53	91	52	11	56	2	9	>53
53	91	52	10	56	0	10	>53

The absolute approximation ratio of FF was improved from 1.75 to $\frac{12}{7} \approx 1.7143$. The obvious open problem here is what is the absolute approximation ratio of FF? Is it as low as the 1.7 lower bound? We conjecture that $\frac{12}{7}$ is not the correct answer. The only values of Opt for which we did not exclude the approximation ratio $\frac{12}{7}$ are numbers between 7 and 63 which are divisible by 7 (i.e., 7, 14, 21, 28, 35, 42, 49, 56, and 63).

An additional well-known algorithm is Best-Fit (BF) [7]. This algorithm places an item in a most full open bin in which it fits among the already open bins, and otherwise opens a new bin. Finding Best-Fit's (BF) absolute approximation ratio would also be interesting, as would determining if the upper bound $\frac{12}{7}$ also applies to BF. The results in this paper which apply to FF, up through and including Theorem 6, also apply to BF. However, the proofs used for most later results, starting with the assumption that no bin of FF contains two items which are packed in the same bin by Opt, do not appear to necessarily hold for BF. In particular, already Remark 7 does not hold for Best-Fit, and there exist instances where two items from a single Opt bin are packed into in the same BFbin, but if one merges the two items into a single item, the packing of other items by BFis affected, unlike the case with FF. Consider the list $\frac{1}{2} + \varepsilon, \frac{1}{2} + 2\varepsilon, \frac{1}{4} - \varepsilon, \frac{1}{4}, \frac{1}{2} - 2\varepsilon$, where in an optimal packing the first, third, and fourth items are packed in one bin, and the second and fifth items are packed into another bin. BFpacks the first and fourth items in the first bin, the second and third in the second bin, and the fifth in a third bin, whereas if the first and fourth items are merged as suggested in Remark 7, and BFis applied, an optimal packing is obtained.

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